A Quantum Formulation of Game Theory

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Quantum game theory, a subset of quantum information science, is the extension of classical game theory from the perspective of quantum algorithms. In classical game theory, strategies are typically restricted to those that are pure (deterministic) or mixed (probabilistic), but with the introduction of quantum strategies, players, and states, these quantum games exhibit exceptional behavior not present in their classical counterparts.

I. INTRODUCTION

Game theory, a subset of the expansive field of information theory, is the study of mathematical models of strategic interaction between two or more individuals. A number of models that arise from game theory have numerous rich applications to a range of fields, including logic, computer science, economics, biology, and sociology [8].

Traditionally, classical game theory has dealt with rational decision-makers (**players**) and their options that they choose in a setting where the outcome of depends not only on their own actions, but also on the actions of others (**strategies**) [5, 8]. These options are selected in such a way that the players' **utilities** are maximized, so that each player strives to get the largest payoff possible. Informally, utility represents the motivations of the players; a utility function assigns a real number to every possible outcome of the game, with higher utility denoting a more preferable outcome. With the introduction of strategies involving quantum superpositions of regular strategies (**quantum strategies**) and states (**quantum states**), a number of results in the classical formulation of game theory can be extended [4].

Consider the *prisoner's dilemma*, a famous two-player game from classical game theory. In this game, two prisoners Alice and Bob have two choices: Cooperate (C) or Defect (D). If both prisoners defect, then each prisoner will serve two years in prison—both players will end up with a utility of -2. If one prisoner cooperates and the other defects, then the one who cooperates is set free and the other will serve *three* years in prison. Finally, if both prisoners cooperate, then each prisoner will serve *one* year in prison. This game can be modeled using a *payoff matrix*, a matrix whose entries are the utilities of the row (Alice) and column (Bob) players, respectively.

	C	D
C	$-1, -1$	$-3, 0$
D	$0, -3$	$-2, -2$

Assuming that Alice and Bob are perfectly rational, neither will prefer to cooperate over defect. Indeed, if Alice cooperates, then Bob has payoff -1 if he cooperates and payoff 0 if he defects, so Bob will always want to defect in this case. If Alice defects, then Bob has payoff -3 if he cooperates and payoff -2 if he defects, so again Bob will always want to defect. By symmetry, Alice will want to defect no matter what Bob plays. Therefore, both Alice and Bob will choose to defect, resulting in payoffs -2 for both players. Even though it is in their best interests to both cooperate, resulting in payoffs of -1, two perfectly rational individuals will both defect in a single instance of this game [8].

When the notion of strategies is extended to the quantum domain using qubits to represent players' states, even simple games like the prisoner's dilemma are endowed with a mathematically rich and strategically complex structure. In the case of the *quantum* prisoner's dilemma, it turns out that we can parametrize the "entanglement" of a continuum of different games using our quantum formulation, and the solutions to these games will differ depending on the value of our parameter [1, 4, 7].

In this paper, we first give a quantum formulation of quantum game theory analogous to the classical formulation of classical game theory, and we then discuss the properties, structures, and solutions to quantum games that arise when we extend their classical counterparts to the quantum domain.

II. CLASSICAL GAME THEORY

A. Classical Formulation

We first give a brief overview of the notion of strategy in classical game theory. An **information set** of a player is a set that establishes all possible moves that could have taken place in the game so far. That is, when a player is in a certain information set, he/she knows that one of the nodes in the information set is reached, but she cannot rule out any of the nodes in the information set [8].

A **strategy** of a player is a complete contingent-plan that determines which action they will take at each information they are to move. Equivalently, a strategy of a player i is a function s_i that maps every information set h_i of player i to an action that is available at h_i. A **strategy profile** of a game with players $N = \{1, 2, \dots, n\}$ is a vector $s = (s_1, \dots, s_n)$ of strategies. The set of all strategy profiles is denoted by $S = S_1 \times \dots S_n$, where S_i is the set of all strategies of player i [8].

In 1944, the mathematician John von Neumann introduced the concept of a strategy profile where no player can do better by unilaterally changing his or her strategy. This would later be known as a **Nash equilibrium**, named after the mathematician John Nash after his publication of his famous 1951 article "Non-Cooperative Games" [6]. In other words, a Nash equilibrium is a strategy profile in which no player has an incentive to deviate if all the other players follow the strategies that are prescribed to them [5].

Formally, a strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ is a Nash equilibrium if s_i^* is a best response to $s_{-i}^* = (s_1^*, \ldots, s_{i-1}^*, s_{i+1}^*, \ldots, s_n^*)$ for each i. That is, for each i,

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \quad \forall\ s_i \in S_i.$$

Furthermore, Nash equilibria can be in both *pure* and *mixed* strategies. A **pure strategy** is one that provides a complete definition of how a player will play a game. A **mixed strategy** is one that assigns a probability to each pure strategy, and allows for the player to randomly select a pure strategy.

In some games, there may be a number of Nash equilibria, so there are many different ways to narrow down the number of solutions (known as *refinements*), which generally depend on the type of game. Prominent examples of these refinements include *subgame perfect Nash equilibrium* (which arise in repeated games), *Bayesian Nash equilibrium* (which arise in games of incomplete information), and *sequential equilibrium* (which arise in extensive form games).

Arguably the most famous theorem in classical game theory is the one that won John Nash his Nobel Prize in Economics. Define a **finite** game to be one with a finite number of players in which each player can choose from finitely many strategies. **Nash's Existence Theorem** then states that *every finite game has at least one Nash equilibrium in pure or mixed strategies* [5, 8].

Another famous theorem in classical game theory is John von Neumann's *minimax theorem*, which has seen a number of applications in economics and computer science. Let P and Q be mixed strategies for Alice and Bob, and let A be their payoff matrix. Then the **Minimax Theorem** states that

$$\max_P \min_Q \mathbb{E}[P, Q] = \min_Q \max_P \mathbb{E}[P, Q] = v$$

where $\mathbb{E}[P, Q] = P^T A Q$ is the *expected payoff.* Here, v is said to be the **value** of the game. As a consequence of the Minimax Theorem, there must always exist optimal strategies P^*, Q^* [8].

Finally, we say that a game is **zero-sum** if each player's gain or loss of utility is exactly balanced by the losses and gains of the other participants. An example of a two-player zero-sum coordination game is

	A	B
A	$-a, a$	$b, -b$
B	$c, -c$	$-d, d$

B. Classical Games

Let us return to the example of the prisoner's dilemma. In the example given above, the only pure-strategy Nash equilibrium is (D, D); both players will receive payoff -2, but neither is incentivized to deviate. There is no mixed-strategy Nash equilibrium. If there were, then we could write the strategy profile $(pC + (1-p)D, qC + (1-q)D)$ with probabilities $p, q \in (0, 1)$ for Alice and Bob, respectively. For Alice, the expected utility of playing C is

$$u_A(C, q) = q u_A(C, C) + (1-q) u_A(C, D) = 2q - 3$$

and the utility of playing D is

$$u_A(D, q) = q u_A(D, C) + (1-q) u_A(D, D) = 2q - 2.$$

Since Alice's payoff of playing D is always greater than playing C, she is always incentivized to play D, so the above strategy profile cannot be a mixed Nash equilibrium, as desired.

Some games that do not have a Nash equilibrium in pure strategies may have an equilibrium in mixed strategies. Consider the game of *hide and seek*, given by the following payoff matrix:

	U	D
U	$1, 0$	$0, 1$
D	$0, 1$	$1, 0$

This is another example of a two-player coordination game with perfect information. Notice that there is no pure-strategy Nash equilibrium; no matter what choice a player chooses, they are always incentivized to choose the other. However, there is a *mixed-strategy* Nash equilibrium: $(\frac{1}{2}U + \frac{1}{2}D, \frac{1}{2}U + \frac{1}{2}D)$. Indeed, each player receives payoff $\frac{1}{2}$ according to this strategy profile, and it can be easily checked that any deviation will result in a lower payoff for each player [8].

III. QUANTUM GAME THEORY

A. Motivation

We now extend the domain of classical game theory, in which we considered only pure and mixed strategies, to that of quantum game theory, in which the states and strategies have quantum analogs in quantum states (both pure and mixed) and quantum strategies.

Let Ω be a sample space, σ be the set of events, and p be a probability distribution over σ. Formally, in classical game theory, we can think of **players** to be simple random variables $X : (\Omega, \sigma, p) \to \Lambda$, where (Ω, σ, p) is a probability space with finite cardinality and $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ are the *moves* of X [2]. Since

$$\sum_{i=1}^{n} p(X = \lambda_i) = 1,$$

we note that Λ can be put in one-to-one correspondence with some subset of $\mathbb{R}$. This motivates us to consider the quantum formulation as one where the moves are eigenvalues of some self-adjoint operator in a Hilbert space.

We can associate X with an $n \times n$ self-adjoint matrix given by $X = (x_{ij})$, where $x_{ii} = X(i)$ and $x_{ij} = 0$ if $i \neq j$ [2]. Then player X corresponds to the spectral resolution

$$X = \sum_{i=1}^{n} \lambda_i E_i$$

where $\lambda_i = X(i)$ and E_i is the one-dimensional projection matrix with $(E_i)_{ii} = 1$ and $(E_i)_{ij} = 0$. Furthermore, if we let $\rho = (\rho_{ij})$ to be the $n \times n$ matrix with $\rho_{ii} = p(i)$ and $\rho_{ij} = 0$, then ρ is a positive matrix with unit trace, or a **state**; we also have $\operatorname{tr} \rho E_i = p_i$.

Similarly, for two-player quantum games, we can think of the players X and Y as self-adjoint operators $X : H_1 \to H_1$ and $Y : H_2 \to H_2$ with spectral resolutions

$$X = \sum_{i=1}^{n} \lambda_i E_i$$

and

$$Y = \sum_{i=1}^{k} \mu_j F_j,$$

where $(H_1, P(H_1), \rho_1)$ and $(H_2, P(H_2), \rho_2)$ are two finite-dimensional quantum probability spaces. We say that players X and Y have available moves $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_k$, respectively [2]. Thus, $\operatorname{tr} \rho_1 E_i$ is the probability that X will make move λ_i, and $\operatorname{tr} \rho_2 F_j$ is the probability that Y will make move μ_j.

B. Quantum Analogs of Classical Game Theory

Using the sets of possible moves (λ_i) and (μ_j), we can find a payoff matrix A, analogous to that in classical game theory. Define the expected payoff of the **strategy profiles** $\mathcal{E} = (E_1, \ldots, E_n)$ and $\mathcal{F} = (F_1, \ldots, F_k)$ by

$$\begin{aligned}\mathbb{E}[\mathcal{E}, \mathcal{F}] &= \operatorname{tr} \rho_1 \mathcal{E} \cdot A \cdot \operatorname{tr}(\rho_2 \mathcal{F})^T \\ &= \sum_{i=1}^{n} \sum_{j=1}^{k} \operatorname{tr} \rho_1 E_i a_{ij} \operatorname{tr} \rho_2 F_j.\end{aligned}$$

If $n = k$ and $\rho_1 = \rho_2 = \rho = |u\rangle\langle u|$, where $u \in \mathbb{C}^n$ is a unit vector and $|u\rangle\langle u|\,(\omega) = \langle u, \omega | u, \omega\rangle\, v$, then ρ is a pure state and $\mathcal{E}$ commutes with $\mathcal{F}$. That is, $E_i F_j = F_j E_i$ for all i, j. Moreover, $\operatorname{tr} \rho(E_i F_j) = \operatorname{tr} \rho(E_i) \cdot \operatorname{tr} \rho(F_j)$, so $\mathbb{E}[\mathcal{E}, \mathcal{F}]$ can be thought of as the total expected payoff of a classical two-person zero-sum game between independent players X and Y.

We can now state the quantum variant of the Minimax Theorem. First, if $(H, P(H), \rho)$ is an n-dimensional quantum probability space and $\mathcal{E} = \{E_1, E_2, \ldots, E_n\} \subset P(H)$ such that E_i is a one-dimensional projection for all i and

$$\operatorname{tr} \rho \left(\sum_{i=1}^{n} E_i - I \right) = 0,$$

then we say that $\mathcal{E}$ is a "ρ-resolution of the $(n \times n)$ identity" [2].

Furthermore, we define an equivalence relation $\equiv$ in the set of all such ordered resolutions of the identity by $\mathcal{E}' \equiv \mathcal{E} \pmod{\rho}$ if and only if $\operatorname{tr} \rho \mathcal{E}' = \operatorname{tr} \rho \mathcal{E}$. Denote this equivalence class corresponding to $\mathcal{E}$ by $[\mathcal{E}]_\rho$. Then the quantum Minimax Theorem can be stated as follows.

Let $\rho \in \mathbb{C}^N$ and $\sigma \in \mathbb{C}^K$ be pure states, and let A be an $(N \times K)$ payoff function. If $\mathcal{E}$ is an N-fold orthogonal resolution of the $(N \times N)$ identity and $\mathcal{F}$ is a K-fold orthogonal resolution of the $(K \times K)$ identity, then

$$\max_{[\mathcal{E}]_\rho} \min_{[\mathcal{F}]_\sigma} \mathbb{E}[\mathcal{E}, \mathcal{F}] = \min_{[\mathcal{F}]_\sigma} \max_{[\mathcal{E}]_\rho} \mathbb{E}[\mathcal{E}, \mathcal{F}].$$

This theorem serves as a quantum analog to the famous classical Minimax Theorem using, formalized within the framework of the spectral theorem for self-adjoint operators on finite-dimensional quantum probability spaces [2].

IV. QUANTUM GAMES

The most fundamental solution concept in game theory is the Nash equilibrium. A large portion of classical game theory deals with variants of classical games and the effects on the Nash equilibria. By extending games into the quantum domain, we see that new Nash equilibria can arise in different ways.

Let us revisit the prisoner's dilemma, but with a quantum twist: we now allow players to adopt quantum strategies. Each player has a qubit and can manipulate it independently in this version of the game. Per our quantum formulation above, we assign possible outcomes of the classical strategies C and D the two basis vectors of a qubit, denoted by $|C\rangle = |\uparrow\rangle = \binom{1}{0}$ and $|D\rangle = |\downarrow\rangle = \binom{0}{1}$. Furthermore, let us adjust the payoff matrix so that the Nash equilibrium stays the same but all payoffs are now nonnegative:

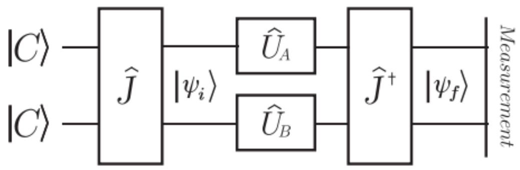

FIG. 1: A quantum circuit for the quantum prisoner's dilemma [4].

	C	D
C	$3,3$	$0,5$
D	$5,0$	$1,1$

Consider the one-parameter gate

$$\hat{J} = \exp\left\{ i\gamma \hat{D} \otimes \hat{D}/2 \right\},$$

where $\gamma \in [0, \pi/2]$. This gate $\hat{J}$ produces entanglement between the two qubits [3, 4]. If the game starts from the pure state $|CC\rangle$, then after passing the state through the gate, the initial state becomes

$$|\psi_i\rangle = \hat{J}\,|CC\rangle = \cos\frac{\gamma}{2}\,|CC\rangle + i\sin\frac{\gamma}{2}\,|DD\rangle\,.$$

The parameter γ can be considered as a measure of the game's *entanglement*.

The physical model of this quantum game is as follows:

After the initial state $|\psi_i\rangle$ is produced, each player applies a unitary transformation on their individual qubit. Subsequently the game's state goes through $\hat{J}^\dagger$ and the final state becomes $|\psi_f\rangle$. According to the corresponding entries of the payoff matrix, we have the utilities

$$u_A = 3P_{CC} + 1P_{DD} + 5P_{DC} + 0P_{CD},$$
$$u_B = 3P_{CC} + 1P_{DD} + 0P_{DC} + 5P_{CD}$$

where $P_{\sigma\sigma'} = |\,\langle\sigma\sigma'|\psi_f|\sigma\sigma'|\psi_f\rangle\,|^2$ is the probability that the final state will collapse into $|\sigma\sigma'\rangle$ [4]. Each player gets payoff according to their payoff function.

Now consider the case where the strategic space is restricted to a two-parameter strategy set as a subset of the whole unitary space. This unitary operator as a function of parameters θ, ϕ is given by

$$U(\theta,\phi) = \begin{pmatrix} e^{i\phi}\cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & e^{-i\phi}\cos\frac{\theta}{2} \end{pmatrix}$$

where $\theta, \phi \in [0, \pi/2]$. We note that

$$U(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is the identity operator, and corresponds to Cooperate, and

$$U(\pi,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is the bit-flip operator, and corresponds to Defect [1, 4, 7].

For the game with $\gamma = 0$, there exists a pair of quantum strategies $\hat{D} \otimes \hat{D}$ (this strategy $\hat{D}$ can be thought of as "quantum Defect"), which is the Nash equilibrium and yields payoff $(1,1)$. This game behaves classically [1, 4, 7], and the Nash equilibrium and final payoffs of the game are equivalent to those in the classical prisoner's dilemma. Such a game is considered to be "minimally entangled."

For the game with $\gamma = \pi/2$, there exists a pair of strategies $\hat{Q} \otimes \hat{Q}$ (this strategy $\hat{Q}$ can be thought of as "quantum Cooperate"), which is a Nash equilibrium and yields payoff $(3,3)$. As it turns out, this result is **Pareto optimal**—in which no individual is made better off without making at least one other individual worse off [4]—and as such the "dilemma" that was present in the original classical prisoner's dilemma game has been eliminated [1, 7]. By allowing quantum strategies in the game, we have managed to find a Nash equilibrium in *quantum strategies* that yields better payoffs for both players and was classically forbidden.

In both the quantum and classical versions of the prisoner's dilemma, in the decision-making step, no player has any information about which strategy the other will adopt. As such, this is a fascinating result that is reminiscent of unintuitive results in quantum information, as a consequence of entanglement of the two qubits. In essence, entanglement serves as a sort of *contract* between the players of the game.

It is natural to ask what happens in quantum prisoner's dilemma games that are neither minimally nor maximally entangled; that is, $\gamma \neq \{0, \pi/2\}$. As it turns out, there are two thresholds of the game's entanglement at

$$\gamma_1 = \sin^{-1}\sqrt{\frac{1}{5}} \quad \text{and} \quad \gamma_2 = \sin^{-1}\sqrt{\frac{2}{5}}.$$

For games with $\gamma \in (0, \gamma_1]$, the quantum game behaves classically, with features identical to the case where $\gamma = 0$ [4]. The Nash equilibrium of the game is again $\hat{D} \otimes \hat{D}$ and the final payoff vector is $(1,1)$.

For games with $\gamma \in [\gamma_1, \gamma_2]$, the quantum game shows features that are present in neither the $\gamma = 0$ nor the $\gamma = \pi/2$ case. Here, $\hat{D} \otimes \hat{D}$ is no longer the Nash equilibrium, but *two* Nash equilibria arise in the form of $\hat{D} \otimes \hat{Q}$ and $\hat{Q} \otimes \hat{D}$. The payoff of the strategy $\hat{D}$ is $5\cos^2\gamma$ and that of $\hat{Q}$ is $5\sin^2\gamma$. Since $\gamma \leq \pi/2$, we have $5\cos^2\gamma \geq 5\sin^2\gamma$, so the player playing $\hat{D}$ will earn a higher payoff. Even though the prisoner's dilemma is fundamentally symmetric, the introduction of quantum states causes a surprising asymmetry to arise.

Finally, for games with $\gamma \in [\gamma_2, \pi/2)$, the quantum game behaves similar to the maximally entangled case, with Nash equilibrium $\hat{Q} \otimes \hat{Q}$ with payoff vector $(3,3)$. As such, depending on the value of the parameter γ, the "dilemma" in the prisoner's dilemma can be continuously "removed" [4].

Note that the discontinuities in this quantum game can be considered as entanglement-correlated phase transi-

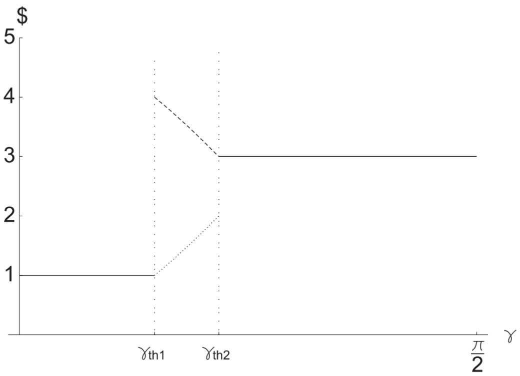

FIG. 2: Expected utility of the quantum prisoner's dilemma as a function of γ [4].

tions. As we continuously move γ along the real line, we can obtain a number of (independently) classical games that yield Nash equilibria different from the original game. To summarize, for $\gamma \in [0, \gamma_1]$, the quantum game yields no advantage over the classical game. For $\gamma \in [\gamma_1, \gamma_2]$, there are *two* Nash equilibria, both of which are *asymmetric*. For $\gamma \in [\gamma_2, \pi/2]$, a *new* Nash equilibrium arises, which is not only *symmetric* but also yields higher payoffs than the original game. A graph of the expected utilities of the quantum prisoner's dilemma as a function of γ is as follows:

Finally, we can also adjust the values of the payoff matrix (while keeping the unique Nash equilibrium at (D, D)) to vary the phase transition behavior. For some payoff values, the transition phase in which the quantum game has two asymmetric Nash equilibria disappears; for others, the quantum game may not even display a transition phase, or even stranger, the classical and quantum phases may overlap and create a "coexistence phase" [4].

V. DISCUSSION

Classical game theory, despite being a relatively new field of mathematics, has been studied extensively by mathematicians. Extending game theory to the quantum domain by introducing quantum strategies and quantum states is an even newer development in the field, and solutions to such quantum games often yield interesting and unexpected results that would never have appeared in their classical variants. In summary, quantum game theory is a promising area of study in the burgeoning field of quantum information that provides a totally new outlook on the purview of classical game theory.

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